Real chaos, and complex time

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Abstract

Real vector fields $\dot{z} = F(z)$ in $\mathbb{R}^N$ extend to $\mathbb{C}^N$, for entire F. We do not impose restrictions on the dimension N. The homoclinic orbit $z(t) = 1 - 3/\cosh^2(t/\sqrt{2})$ of the pendulum $\ddot{z} = z^2 - 1$ is a planar example. Note the double poles at complex times $t/\sqrt{2} = \mathrm{i}\pi/2 + k\pi$, for integer k.
Jürgen Scheurle and others have studied "invisible chaos". One manifestation are exponentially small upper bounds

$$C(\eta)\exp(-\eta/\varepsilon)$$

on homoclinic splittings under discretizations of step size $\varepsilon > 0$, or under rapid forcings of that period. Here $\eta > 0$ should be less than the distance of any complex poles of the homoclinic orbit $z(t)$ from the real axis. However, what if $z(t)$ itself is entire, and η can be chosen arbitrarily large?
We consider connecting orbits $z(t)$ between limiting hyperbolic equilibria $v_\pm$, for real $t \to \pm\infty$. We assume separately nonresonant real unstable eigenvalues, at v_-, and stable eigenvalues, at v_+. Locally, we study the resulting irrational torus flows, in imaginary time. Globally, we conclude the existence of singularities of $z(t)$ in complex time t. In that sense, real connecting orbits are accompanied by finite time blow-up, in imaginary time.